How to Conduct Linear Stability Analysis?

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Our PDE Model

We will use the following PDE in *Schnakenberg* Model:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + \gamma(a - u + u^2 v), \\ \frac{\partial v}{\partial t} &= d\frac{\partial^2 v}{\partial x^2} + \gamma(b - u^2 v), \end{aligned} \tag{1}$$

on $[0, L]$ with zero-flux boundary condition

$$\left.\frac{\partial u}{\partial x}\right|_{x=0,\, L} = \left.\frac{\partial v}{\partial x}\right|_{x=0,\, L} = 0.$$

Stability Analysis of ODEs

We begin by this question: let A be a constant 2×2 matrix, and

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} u \\ v \end{bmatrix} = A\begin{bmatrix} u \\ v \end{bmatrix}.$$

Let λ be an **eigenvalue** of A, and $\vec{q}$ be the corresponding eigenvector. Then $A\vec{q} = \lambda\vec{q}$, and the ODE has solution

$$\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = \vec{q}e^{\lambda t}.$$

The idea of linear stability analysis (LSA) is similar

Stability Analysis of ODEs

Assume we have

$$\begin{aligned} \dot{u} &= f(u, v), \\ \dot{v} &= g(u, v). \end{aligned} \tag{2}$$

And there exists constant (u_0, v_0) such that

$$f(u_0, v_0) = g(u_0, v_0) = 0.$$

How to linearize (2) near (u_0, v_0)?

Stability Analysis of ODEs

Take $\varepsilon > 0$ small, and assume

$$u(t) = u_0 + \varepsilon u_1(t),$$
$$v(t) = v_0 + \varepsilon v_1(t),$$

And taking them back into (2), collecting all $O(\varepsilon)$ terms:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \underbrace{\begin{bmatrix} f_u & f_v \\ g_u & g_v \end{bmatrix}\Bigg|_{(u_0,v_0)}}_{:=J} \begin{bmatrix} u_1 \\ v_1 \end{bmatrix}.$$

Here J is a constant matrix

Stability Analysis of ODEs

Stable manifold theorem says:

- If all the eigenvalues of J have negative real parts, then (u_0, v_0) is **stable**
- If one of the eigenvalues of J has positive real parts, then (u_0, v_0) is **unstable**

How to deal with PDEs

Now assume we have not only (2), but also spatial random walk,

$$\begin{aligned} \frac{\partial u}{\partial t} &= D_u \frac{\partial^2 u}{\partial x^2} + f(u,v), \\ \frac{\partial v}{\partial t} &= \underbrace{D_v \frac{\partial^2 v}{\partial x^2}}_{\text{diffusion}} + \underbrace{g(u,v)}_{\text{reaction}}. \end{aligned} \tag{3}$$

If there is constant (u_0, v_0) such that $f(u_0, v_0) = g(u_0, v_0) = 0$, then

$$(u(t,x), v(t,x)) \equiv (u_0, v_0)$$

is a solution to (3), which is a homogeneous equilibrium

How to deal with PDEs

We want to use the same idea to linearize (3) near (u_0, v_0), we take $\varepsilon > 0$ small and try

$$\begin{aligned} u(t,x) &= u_0 + \varepsilon u_1(t,x), \\ v(t,x) &= v_0 + \varepsilon v_1(t,x). \end{aligned} \tag{4}$$

Again we collect all $O(\varepsilon)$ terms, we have

$$\frac{\partial}{\partial t}\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \underbrace{\begin{bmatrix} D_u & 0 \\ 0 & D_v \end{bmatrix}}_{:=D} \frac{\partial^2}{\partial x^2}\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} + \underbrace{\begin{bmatrix} f_u & f_v \\ g_u & g_v \end{bmatrix}}_{:=J}\begin{bmatrix} u_1 \\ v_1 \end{bmatrix}. \tag{5}$$

How to deal with PDEs

Denote

$$\mathcal{L} := D\frac{\partial^2}{\partial x^2} + J.$$

A square matrix $A \in \mathbb{R}^{n\times n}$ defines a linear map $T(x) = Ax$ from $\mathbb{R}^n$ to $\mathbb{R}^n$. An eigenvector $\vec{q} \neq 0$ and eigenvalue λ satisfy

$$A\vec{q} = \lambda\vec{q}.$$

This means that along the direction of $\vec{q}$, the linear transformation preserves the direction and only changes the magnitude.

Question: $\mathcal{L}$ is also linear, do we have an analogue for $\mathcal{L}$?

How to deal with PDEs

What is an eigenvector of $\mathcal{L}$?

Techniques: if $\eta(x)$ satisfies that $\frac{\partial^2}{\partial x^2}\eta(x) = \mu\eta(x)$ for some $\mu \in \mathbb{R}$, then let $\vec{q}$ be an eigenvector of $\mu D + J$, which is a constant matrix, with eigenvalue λ, then

$$\mathcal{L}\vec{q}\eta(x) = (\mu D + J)\vec{q}\eta(x) = \lambda\vec{q}\eta(x).$$

We call $\vec{q}\eta(x)$ an **eigenfunction** of $\mathcal{L}$

How to deal with PDEs

How to find such $\eta(x)$? On $[0, L]$, with:

- Periodic boundary condition: $\eta(0) = \eta(L)$, then $\eta(x) = a\cos\left(\frac{2\pi k}{L}x\right) + b\sin\left(\frac{2\pi k}{L}x\right)$, and the corresponding eigenvalue is the eigenvalue of $-\left(\frac{2\pi k}{L}\right)^2 D + J$ for $k \in \mathbb{Z}$
- Zero-flux boundary condition: $\eta'(0) = \eta'(L) = 0$, then $\eta(x) = \cos\left(\frac{\pi k}{L}x\right)$, and the corresponding eigenvalue is the eigenvalue of $-\left(\frac{\pi k}{L}\right)^2 D + J$ for $k \in \mathbb{Z}$

How to deal with PDEs

So our perturbation is

$$\begin{bmatrix} u_1(t,x) \\ v_1(t,x) \end{bmatrix} = A(t)\vec{q}\eta(x).$$

Then we obtain the linearized equation

$$\frac{\mathrm{d}A(t)}{\mathrm{d}t} = (\mu D + J)A(t). \tag{6}$$

- $\mu D + J$ stable $\Longrightarrow$ perturbation is stable
- $\mu D + J$ unstable $\Longrightarrow$ perturbation is unstable

Analytical Strategy

Goal: determine when spatial perturbations grow or decay

- Identify equilibria (ODE)
- Establish solution framework
- Linearize near steady state
- Analyze the eigenvalues of the linearized operator (matrix)

LSA for Schnakenberg Model

Back to (1),

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \gamma(a - u + u^2 v),$$
$$\frac{\partial v}{\partial t} = d\frac{\partial^2 v}{\partial x^2} + \gamma(b - u^2 v),$$

We have homogeneous equilibrium

$$u_0 = a + b, \quad v_0 = \frac{b}{(a+b)^2}.$$

LSA for Schnakenberg Model

To observe pattern formation (Turing instability), we always need the equilibrium to be stable in the ODE (reaction) part. Then pattern emerges as spatial movement destabilizes the stable ODE equilibrium. Here

$$J = \gamma \begin{bmatrix} 2u_0v_0 - 1 & u_0^2 \\ -2u_0v_0 & -u_0^2 \end{bmatrix} = \gamma \begin{bmatrix} \frac{b-a}{a+b} & (a+b)^2 \\ -\frac{2b}{a+b} & -(a+b)^2 \end{bmatrix}.$$

We have

$$\mathrm{tr}(J) = \gamma \left[\frac{b-a}{a+b} - (a+b)^2 \right],$$
$$\det(J) = \gamma^2 (a^2 + b^2 + 2ab) = \gamma^2 (a+b)^2 > 0.$$

LSA for Schnakenberg Model

Since we impose zero-flux boundary condition, we assume the following perturbation

$$u(t,x) = u_0 + \varepsilon q_u e^{\lambda t} \cos\left(\frac{\pi k}{L} x\right),$$

$$v(t,x) = v_0 + \varepsilon q_v e^{\lambda t} \cos\left(\frac{\pi k}{L} x\right),$$

where $k \in \mathbb{Z}$ and $\varepsilon > 0$ small

LSA for Schnakenberg Model

Use the same idea, we obtain

$$\lambda \begin{bmatrix} q_u \\ q_v \end{bmatrix} = \Bigg(\underbrace{-\Big(\frac{\pi k}{L}\Big)^2 \begin{bmatrix} 1 & 0 \\ 0 & d \end{bmatrix} + \gamma \begin{bmatrix} 2u_0v_0 - 1 & u_0^2 \\ -2u_0v_0 & -u_0^2 \end{bmatrix}}_{:=M(k)} \Bigg) \begin{bmatrix} q_u \\ q_v \end{bmatrix}.$$

- $M(0)$ is always stable since (u_0, v_0) is stable in ODE
- If for all $k \in \mathbb{Z}$, $M(k)$ is stable, then the perturbation is stable
- If there exists some $k^* \in \mathbb{Z}$ with $k^* \neq 0$ such that $M(k^*)$ is unstable, then k^* is the mode we want to find

LSA for Schnakenberg Model

For Turing instability, we need to choose a parameter, we call it the **bifurcation parameter**. When this bifurcation parameter crosses some threshold, $M(k)$ admits some eigenvalues with positive real parts.

We use d as the bifurcation parameter, and for Turing instability, the threshold is determined by

$$\det(M(k)) = d\left(\frac{\pi k}{L}\right)^4 - \gamma(2u_0v_0d - u_0^2 - d)\left(\frac{\pi k}{L}\right)^2 + \gamma^2 u_0^2 < 0.$$

LSA for Schnakenberg Model

For fixed $k \in \mathbb{Z}$ and $k \neq 0$, we can calculate

$$d > d^*(k) := \frac{\gamma u_0^2 \left[\gamma + \left(\frac{\pi k}{L}\right)^2\right]}{\left(\frac{\pi k}{L}\right)^2 \left[\gamma(2u_0 v_0 - 1) - \left(\frac{\pi k}{L}\right)^2\right]}. \tag{7}$$

We also need

$$\gamma(2u_0 v_0 - 1) - \left(\frac{\pi k}{L}\right)^2 > 0.$$

For emergence of patterns, we usually choose

$$d > d^* := \min_{k \in \mathbb{Z}} d^*(k). \tag{8}$$